AB Calculus Review Sheet #3

Rolle's Theorem: Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). If $f(a) = f(b)$ then there is at least one number c in (a,b) such that $f'(c) = 0$.

The Mean Value Theorem: If f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then there exists a number c in (a,b) such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

The Fundamental Theorem of Calculus: If a function f is continuous on the closed interval $[a,b]$ and F is an antiderivative of f on the interval $[a,b]$, then $\int_a^b f(x)dx = F(b) - F(a)$.

The Second Fundamental Theorem of Calculus: If f is continuous on an open interval I containing a, then for every x in the interval, $\dfrac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$.

Definition of the Average Value of a Function on an Interval: If f is integrable on the closed interval $[a,b]$, then the average value of f on the interval is $\dfrac{1}{b-a}\int_a^b f(x)dx$..

Mean Value Theorem for Integrals: If f is continuous on the closed interval $[a,b]$, then there exists a number c in the closed interval $[a,b]$ such that $\int_a^b f(x)dx = f(c)(b-a)$.

Definition of a Riemann Sum: Let f be defined on the closed interval $[a,b]$ and let Δ be a partition of $[a,b]$ given by $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$, where Δx_i is the length of the th subinterval. If c_i is *any* point in the ith subinterval, then the sum $\sum_{i=1}^{n} f(c_i)\Delta x_i, \quad x_{i-1} \le c_i \le x_i$ is call a Riemann sum of f for the partition Δ.

Definition of a Definite Integral: If f is defined on the closed interval $[a,b]$ and the limit $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$ exists, then f in integrable on $[a,b]$ and the limit is denoted by

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)dx.$$

The limit is called the definite integral of f from a to b. The number a is the lower limit of integration, and the number b is the upper limit of integration.

Multiple Choice Questions:

1. The function $f(x) = x^{2/3}$ on $[-8,8]$ does not satisfy the conditions of the mean-value theorem because

   (A) $f(0)$ is not defined (B) $f(x)$ is not continuous on $[-8,8]$ (C) $f'(-1)$ does not exist

   (D) $f(x)$ is not defined for $x < 0$ (E) $f'(0)$ does not exist

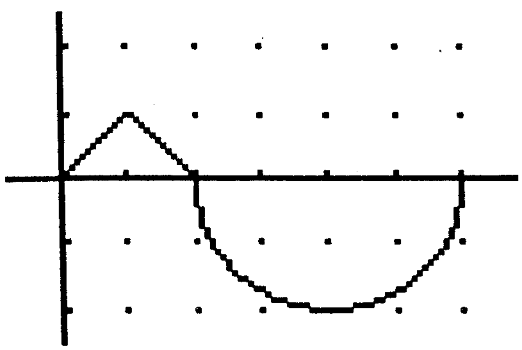
2. If $f(a) = f(b) = 0$ and $f(x)$ is continuous on $[a,b]$, then

   (A) $f(x)$ must be identically zero. (B) $f'(x)$ may be different from zero for all x on $[a,b]$.

   (C) there exists at least one number c, $a < c < b$, such that $f'(c) = 0$.

   (D) $f'(x)$ must exist for every x on (a,b). (E) none of the preceding is true.

Use the graph of $f(x)$ drawn below for questions 3 through 5.



3. $f'(x) = 0$ for $x =$

   (A) 1 only (B) 2 only (C) 4 only (D) 1 and 4 (E) 2 and 6

4. $f'(x)$ does not exist for $x =$

   (A) 1 only (B) 2 only (C) 1 and 2 (D) 2 and 6 (E) 1, 2, and 6

5. $f'(5) =$

   (A) $\frac{1}{2}$ (B) $\frac{1}{\sqrt{3}}$ (C) 1 (D) 2 (E) $\sqrt{3}$

6. A particle starting at rest at $t = 0$ moves along a line so that its acceleration at time t is $12t$ ft/sec^2. How much distance does the particle cover during the first 3 sec?

   (A) 16 ft (B) 32 ft (C) 48 ft (D) 54 ft (E) 108 ft

7. The equation of the curve whose slope at point (x,y) is $x^2 - 2$ and which contains the point $(1,-3)$ is

(A) $y = \frac{1}{3}x^3 - 2x$ (B) $y = 2x - 1$ (C) $y = \frac{1}{3}x^3 - \frac{10}{3}$

(D) $y = \frac{1}{3}x^3 - 2x - \frac{4}{3}$ (E) $3y = x^3 - 10$

8. $\dfrac{d\int_0^t \sqrt{x^3+1}\,dx}{dt} =$

(A) $\sqrt{t^3+1}$ (B) $\dfrac{\sqrt{t^3+1}}{3t^2}$ (C) $3x^2\sqrt{x^3+1}$

(D) $\frac{2}{3}\left(t^3+1\right)\left(\sqrt{t^3+1}-1\right)$ (E) none of these

9. $\dfrac{d}{dx}\int_{\pi/2}^{x^2} \sqrt{\sin t}\,dt =$

(A) $\sqrt{\sin t^2}$ (B) $2x\sqrt{\sin x^2} - 1$ (C) $\sqrt{\sin x^2} - 1$

(D) $\frac{2}{3}\left(\sin^{3/2}(x^2) - 1\right)$ (E) $2x\sqrt{\sin(x^2)}$

10. $\displaystyle\lim_{n\to\infty}\frac{1}{n}\left(1 + \frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \frac{1}{1+\frac{3}{n}} + \cdots + \frac{1}{1+\frac{n-1}{n}}\right) =$

(A) $\int_1^2 \ln x\,dx$ (B) $\int_1^2 \frac{1}{x}\,dx$ (C) $\int_1^2 \frac{1}{1+x}\,dx$

(D) $\int_1^2 x\,dx$ (E) none of these

11. During the worst 4-hour period of a hurricane the wind velocity in mph is given by $v(t) = 5t - t^2 + 100,\ 0 \le t \le 4$. The average wind velocity during this period, in mph, is

(A) 10(B) 100 (C) 102 (D) $104\frac{2}{3}$ (E) $108\frac{2}{3}$

12. Suppose the current world population is 6 billion and the population t years from now is estimated to be $P(t) = 6e^{0.024t}$. Based on this supposition, the average population of the world, in billions, over the next 25 years will be approximately

(A) 6.8 (B) 7.2 (C) 7.8 (D) 8.2 (E) 9.0

13. If $f(x)=\begin{cases} x+1 & \text{for } x<0 \\ \cos(\pi x) & \text{for } x\ge 0 \end{cases}$ then $\int_{-1}^{1} f(x)dx$ is

(A) $\frac{1}{2}+\frac{1}{\pi}$ (B) $-\frac{1}{2}$ (C) $\frac{1}{2}-\frac{1}{\pi}$ (D) $\frac{1}{2}$ (E) $-\frac{1}{2}+\pi$

Free Response Questions:

1. A car, moving with an initial velocity of 5 mph, accelerates at a rate of 2.4 mph per second for 8 seconds.
   a. How fast is the car going when the 8 seconds are up?
   b. How far did the car travel during those 8 seconds? Hint: Change the 8 seconds to $\frac{8}{3600}$ hours then work in hours.

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2. A particle travels with velocity $v(t)=(t-2)\sin t$ meters/sec for $0\le t\le 4$ seconds.
   a. What is the particle's displacement?
   b. What is the total distance traveled by the particle?

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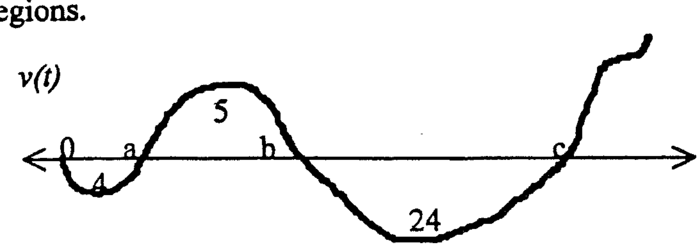
3. A particle moves along the x-axis (units in cm.). Its initial position at $t=0$ seconds is $x(0)=15$ cm. The figure shows the graph of the particle's velocity $v(t)$. The numbers represent the areas of the enclosed regions.



   a. What is the particle's displacement between $t=0$ and $t=c$?
   b. What is the total distance traveled by the particle in the same time period.
   c. Give the positions of the particle at times a, b, and c.
   d. Approximately where does the particle achieve its greatest positive acceleration on the interval $[0,b]$?
   e. Approximately where does the particle achieve its greatest positive acceleration on the inter $[0,c]$?

Review Sheet #3

1. E	6. D	11. D
2. B	7. D	12. D
3. C	8. A	13. D
4. E	9. E	
5. B	10. B	

1. See attached
2. See attached
3. See attached

Review Sheet #3 Solutions

1. This problem mixes the units of time since the acceleration is given in miles per hour per second. To simplify the situation, convert the units of acceleration to miles per hour per hour and the time from seconds to hours.

You are given that. $a(t) = 2.4\dfrac{\text{miles}}{\text{hour}\times\text{second}}$. Convert the seconds to hours by multiplying the by $3600\dfrac{\text{seconds}}{\text{hour}}$. This changes the equation for acceleration to $a(t) = 2.4\times 3600\dfrac{\text{miles}}{\text{hour}\times\text{hour}}$. This means that the change in time (dt) is also in hours so the units of time must be converted too. Change the 8 seconds to $\dfrac{8}{3600}$ hours.

(a) Given $v_0 = 5$ mph and $a(t) = 2.4\times 3600\dfrac{\text{miles}}{\text{hour}\times\text{hour}}$.

$v(t) = \int 3600\times 2.4 dt$. This puts the velocity in miles per hour.

$v(t) = 3600\times 2.4t + C$. Now since $v_0 = 5$ mph $C = 5$ mph and

$v(t) = 3600\times 2.4t + 5$. Remember that t is now in hours rather than seconds.

$v\left(\dfrac{8}{3600}\right) = 3600\times 2.4\left(\dfrac{8}{3600}\right) + 5 = 24.2$ mph. The car is traveling at 24.2 mph after 8 seconds.

(b) Find the total distance traveled in 8 seconds

$$s\left(\frac{8}{3600}\right) = \int_0^{8/3600}(5 + 3600\times 2.4t)\,dt \quad \text{(This is now in miles)}$$

$$s\left(\frac{8}{3600}\right) = 5\left(\frac{8}{3600}\right) + 3600\times 1.2\left(\frac{8}{3600}\right)^2$$

$$s\left(\frac{8}{3600}\right) = 5\left(\frac{8}{3600}\right) + 3600\times 1.2\left(\frac{8}{3600}\right)^2 = .0324444\ldots \text{ miles or } 171.307 \text{ feet.}$$

Review sheet #3
question #2

(2) $v(t) = (t-2)\sin t \quad 0 \le t \le 4$

<u>with a calcutator</u>

(a)
$$\text{disp} = \int_0^4 (t-2)\sin t \, dt$$
$$= -1.450 \text{ meters}$$

The particle is 1.450 meters to the left of the starting location.

(b) Total distance $= \int_0^4 |(t-2)\sin t| \, dt = 1.914$ meters

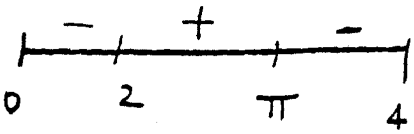
<u>without a calculator</u>

(a) $\text{disp} = \int_0^4 (t-2)\sin t \, dt \qquad u = t-2 \quad dv = \sin t \, dt$

$du = dt \qquad v = -\cos t$

$$\text{Disp} = -(t-2)\cos t + \int \cos t \, dt \Big|_0^4$$
$$= -(t-2)\cos t + \sin t \Big|_0^4$$
$$= (-(2)\cos 4 + \sin 4) - (2\cos 0 + \sin 0)$$
$$= -2\cos 4 + \sin 4 - 2 \quad = -1.450$$

Total distance



$(t-2)\sin t = 0$

$t = 2 \quad \sin t = 0$

$0, \pi$

$$-\int_0^2 v(t)\,dt + \int_2^{\pi} v(t)\,dt - \int_{\pi}^4 v(t)\,dt$$

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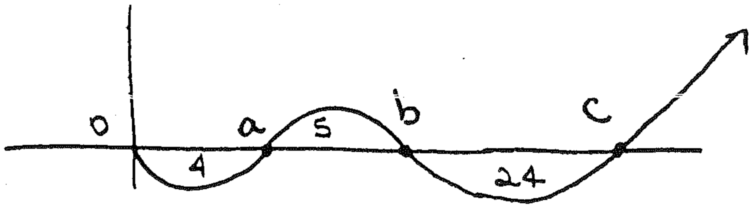

Review Sheet #3
question 3

(a) Disp= $\int_0^c v(t)dt = -4+5-24 = -23$ cm

The particle is 23 cm to the left of its starting position.

b) Total distance = $\int_0^c |v(t)|dt = 4+5+24 = 33$ cm

The particle has traveled a total of 33cm.

c) The position at

$t=0$; 15cm

$t=a$; $15-4 = 11$cm

$t=b$; 11cm+5cm=16cm

$t=c$; 16cm−24cm= −8cm

c) Since $a(t)$ represents to slope of $v(t)$; $a(t)$ will be greatest when $v(t)$ has the largest positive slope for $0<t<b$. This is at $t=a$ when v changes concavity.

d) On the interval $[0,c]$ the max acceleration is at $x=a$ or at $x=c$. The slope at a is greater than the slope at $x=c$; therefore the greatest acceleration is still at $x=a$.